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TWO-DIMENSIONAL INTERACTION OF RIEMANN COMPRESSION WAVES†

V. A. KUKUSHKIN
Ekaterinburg

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A class of solutions of the gas-dynamics equations containing a function arbitrariness is used for a qualitative and quantitative analysis of the gas flow which occurs as a result of the interaction between Riemann compression waves. Two types of flow are investigated matched flow, when the adiabatic exponent is matched in a special way with the initial geometry of the compressed volume, and the general case when there is no such matching. For matched interaction of non-self-similar Riemann waves, a phenomenon of partial collapse is established (only part of the initial mass of the gas is compressed to a point); here the asymptotic estimates for the velocity, density, internal energy and optical thickness are the same as for self-similar compression. It is proved that unmatched interaction of self-similar Riemann waves does not lead to unlimited unshocked compression of the gas; in this case a shock wave occurs when the maximum density of the gas is finite. The results obtained enable us to say that two-dimensional processes of unlimited compression are stable for a fairly wide range of perturbations. © 1999 Elsevier Science Ltd. All rights reserved.

An investigation of the properties of unlimited unshocked compression processes is related, in particular, to the question of whether it is possible to use them to produce the conditions necessary for a thermonuclear reaction to occur [1].

The simplest process of unlimited compression—self-similar compression of a plane layer of gas into a plane—is described by a self-similar Riemann wave [1]. The use of non-self-similar Riemann waves enables one to construct the compression of a plane layer which ensures an unlimited increase in the gas density only on the surface of the compressing piston. In this case the energy costs for such compression may be finite [2].

The flow obtained as a result of matched interaction of self-similar Riemann waves has been constructed analytically [3], for which estimates of the values of the velocity, energy and density [3] and the optical thickness [4] are known. This problem is a special case of the problem of the matched interaction of non-self-similar Riemann waves [5]. To construct an analytic solution, a class of exact solutions of the gas-dynamics equations, namely, double waves, has been used [6]. For the non-self-similar case a two-parameter class of compressing piston control has been investigated. For the flows considered estimates have been derived for the increase in the velocity and density of the gas, which agree with estimates obtained for the self-similar case. It has been proved that part of the initial gas volume is compressed into a certain surface [5].

The purpose of this paper is to investigate a wider class of flows compared with those considered previously.

The fact that a shock wave can occur has been proved analytically for unmatched interaction of self-similar Riemann waves. The gas flow is constructed in the class of solutions with a degenerate hodograph of the double self-similar wave type. The integration of the gas-dynamics equations is reduced to finding a solution of one partial differential equation for one function of two variables (the double-wave equation), which is solved numerically by the method of characteristics.

A class of control of the compressing piston which contains a function arbitrariness has been investigated for coordinated interaction between non-self-similar Riemann waves. For this problem an analytic solution exists [5], but the asymptotic properties of this solution are not obvious. Asymptotic estimates of the gas-dynamics quantities have been obtained by a qualitative analysis of the equations of motion of the gas particles, and the existence of regions of the initial gas volume which are compressed into a surface and into a line was proved. The form of these regions was obtained approximately by numerical integration of ordinary differential equations.

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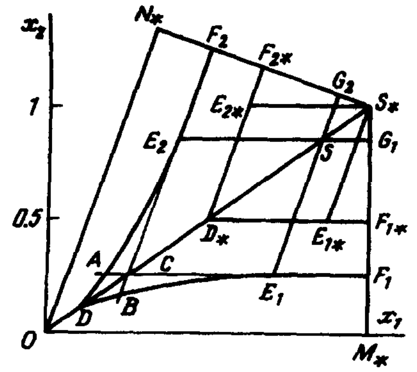

Fig. 1.

1. FORMULATION OF THE PROBLEM

A uniform ideal gas at the instant of time $t = 0$ is at rest inside a prism with a cross-section $OM_*S_*N_*$ (Fig. 1). Consider the equation of state $p = A\rho^{\gamma}$, $1 < \gamma < 3$. Without loss of generality we will assume that the velocity of sound and the density of the unperturbed gas are equal to unity. The sections OM_* and ON_* are the initial position of the compressing piston, which at a certain instant $t < 1$ take the form DE_1F_1 and DE_2F_2. In the trapezia $E_1F_1G_1S$ and $E_2F_2G_2S$ the gas flows are one-dimensional simple waves (non-self-similar Riemann waves). The region of interaction of the one-dimensional flows is denoted by DE_1SE_2, where the solution is constructed in the class of double non-self-similar waves. The unperturbed gas is situated in the quadrilateral $SG_1S_*\, G_2$. The length of the walls M_*S_* and N_*S_* is equal to unity, and at the final instant of time the sound perturbation arrives at the point S_*. We will denote the value of the angle N_*OM_* by 2α. In the case of matched interaction between simple waves, $\alpha = \alpha_0$, where

$$\operatorname{ctg}\alpha_0 = h = \sqrt{\frac{\gamma+1}{3-\gamma}} \tag{1.1}$$

Here, as a result of non-self-similar compression at the final instant $t_* = 1$, the gas occupies the region $D_*E_{1*}S_*E_{2*}$. The points E_{1*} and E_{2*} can be as close to the point S_* as desired. For the self-similar case the region $D_*E_{1*}S_*E_{2*}$ contracts to the point S_*.

2. FLOWS OF GAS OF THE SIMPLE- AND DOUBLE-WAVE TYPE

The motion of the plane parts of the piston (sections E_1F_1 and E_2F_2) is governed by the equations

$$x_2 = f_1(t),\; x_1\sin 2\alpha - x_2\cos 2\alpha = f_2(t).$$

We will assume that the velocities of motion of the plane parts of the piston at the initial instant are equal to zero, and that they then increase monotonically. We will introduce the following notation: U_1 and U_2 are the components of the velocity vector, c is the velocity of sound, $\Theta = \sigma c$, $\sigma = 2/(\gamma-1)$, $(f_i^1)^{-1} = z_i$ are inverse functions for the derivatives f_i^1. The gas flow is given by the following relations [6]

$$u_1'x_1 + u_2'x_2 - \left(\sigma^{-1}\Theta + u_1'u_1 + u_2'u_2\right)t + \Delta_i(\Theta),\; u_i' = \frac{du_i}{d\Theta}$$

In region $E_1F_1G_1S$

$$u_1 = 0,\; u_2 = \Theta - \sigma \tag{2.1}$$

In region $E_2F_2G_2S$

$$u_1 = (\Theta - \sigma)\cos\alpha,\; u_2 = -(\Theta - \sigma)\sin\alpha \tag{2.2}$$

The functions $\Delta_i(\Theta)$ can be found from the condition that a simple wave should be attached to the piston [5]

$$\Delta_i(\Theta)=\left(\frac{\gamma+1}{2}\Theta-\sigma\right)y_i-f_i(y_i),\ \ y_i=z_i(\Theta-\sigma)$$

In the self-similar case the functions $\Delta_i(\Theta)$ are constant.

To describe the motion in the region DE_1SE_2 we need to solve the equations of the non-self-similar double waves for the functions $\Theta=\Theta(u_1,u_2)$ and $\chi=\chi(u_1,u_2)$ [6]

$$\Theta\left(\left(1-\Theta_2^2\right)\Theta_{11}+2\Theta_1\Theta_2\Theta_{12}+\left(1-\Theta_1^2\right)\Theta_{22}\right)+(1-\sigma)\left(\Theta_1^2+\Theta_2^2\right)+2\sigma=0 \tag{2.3}$$

$$\left(1-\Theta_1^2\right)\chi_{22}+2\Theta_1\Theta_2\chi_{12}+\left(1-\Theta_2^2\right)\chi_{11}=0 \tag{2.4}$$

$$\left(\Theta_i=\frac{\partial\Theta}{\partial u_i},\ \Theta_{ij}=\frac{\partial^2\Theta}{\partial u_i\partial u_j},\ \chi_i=\frac{\partial\chi}{\partial u_i},\ \chi_{ij}=\frac{\partial^2\chi}{\partial u_i\partial u_j}.\right)$$

It follows from (2.1) and (2.2) that for gas particles in the region DE_1SF_2, the velocity vector (u_1,u_2) belongs to the set $0\leqslant u_1<\infty$, $u_2>-u_1\text{ctg}2\alpha$. The implicit formulae

$$x_i=(u_i+\sigma^{-1}\Theta\Theta_i)t+\chi_i \tag{2.5}$$

specify the gas flows in the double-wave region. The boundary conditions for the functions $\Theta=\Theta(u_1,u_2)$ and $\chi=\chi(u_1,u_2)$ correspond to the condition that the double wave should join the simple wave. For interaction between two similar simple waves, the gas flow will be symmetrical about the straight line OS_*, and in this case it is sufficient to obtain the solution in the region DE_1S. The condition that the two waves of different ranks should join is satisfied on the section E_1S, and the impermeability condition $u_1=u_2\text{ctg}\alpha$ is satisfied on section DS.

3. THE MATCHED CASE

If the value of the angle α is given by relation (1.1), the solution of Eq. (2.3) has the form

$$\Theta=\sigma+hu_1+u_2 \tag{3.1}$$

We will write the general solution of Eq. (2.4) in the form [5]

$$\chi=\Psi_1(u_1)+\Psi_2(\eta),\ \ \eta=u_1+\delta u_2,\ \ \delta=\sqrt{(\gamma+1)(3-\gamma)}/(\gamma-1)$$

The derivatives $\Psi_1'(u_1)$ and $\Psi_2'(\eta)$ are found from the condition for the double wave to join the simple waves [5]

$$\Psi_1'(u_1)=-\kappa\left\{(1+hu_1)z_2(\kappa u_1)-f_2\left[z_2(\kappa u_1)\right]\right\}$$

$$\Psi_2'(\eta)=-\frac{1}{\delta}\left\{\left(1+\frac{\gamma+1}{2}\frac{\eta}{\delta}\right)z_1\left(\frac{\eta}{\delta}\right)-f_1\left[z_1\left(\frac{\eta}{\delta}\right)\right]\right\},\ \ \kappa=\frac{\sigma}{\delta}$$

Hence, the gas flow in region DE_1SE_2 is given by the formulae

$$x_1=r_*h+\left(u_1+\sigma^{-1}(\sigma+hu_1+u_2)h\right)\tau+g_1(u_1)+\delta^{-1}g_2(\eta) \tag{3.2}$$

$$x_2=r_*+\left(1+\frac{h}{\sigma}u_1+\frac{\gamma+1}{2}u_2\right)\tau+g_2(\eta)$$

$$g_1(u_1)=\Psi_1'(u_1)-r_*h-\frac{t_*-r_*}{\delta}+\left(h+\frac{3}{3-\gamma}u_1\right)t_*$$

$$g_2(\eta) = \delta\Psi_2'(\eta) - r_* + \left(1 + \frac{h}{\sigma}\eta\right)t_*$$

Here $\tau = t - t_*$, r_* is the length of the sections M_*F_{1*} and N_*F_{2*} (Fig. 1). The parameter r_* gives the position of the rectilinear parts of the piston (the sections E_1F_1 and E_2F_2) at the final instant of time t_*.

The laws of motion of the plane parts of the piston (the functions f_1 and f_2) determine the functions g_1 and g_2. We will consider Riemann waves in which the gradient catastrophe only occurs at the final instant of time; then the gradient catastrophe does not occur in the region of interference between one-dimensional flows up to the final instant [5]. In addition, the gas density on the piston surface at the final instant must be infinite. If these limitations are satisfied, g_1 and g_2 do not increase in the interval $[0, +\infty)$, $g_2(0) = 1 - r_*$, $g_1(0) = h(1 - r_*)$, $g_1(\infty) = g_2(\infty) = 0$. For the case of self-similar compression $g_1 = g_2 = 0$, $r_* = 1$. The particle trajectories are found from the equation

$$dx_i/d\tau = u_i(x_1, x_2, \tau),\ i = 1, 2 \tag{3.3}$$

Formulae (3.2) implicitly define the function $u_i(x_1, x_2, \tau)$, and the absence of a gradient catastrophe ensures that Eqs (3.2) are solvable for the velocity components [5]. The class of solutions considered contains function arbitrariness, the functions g_1 and g_2.

4. MATCHED INTERACTION BETWEEN NON-SELF-SIMILAR RIEMANN WAVES

It follows from the fact that the functions g_1 and g_2 are monotonic, that the functions $x_i(u_1, u_2, \tau)$, $i = 1, 2$, given by (3.2), for fixed τ, decrease with respect to u_1 and u_2 (the property of monotonic decrease).

Assertion 1. Consider a mobile region, bounded by the condition

$$x_2 \leq f_1(\tau) \tag{4.1}$$

The gas particles to be found in region (4.1) at a certain instant of time $\tau_1 \in [-1, 0)$, when $\tau > \tau_1$ will remain in this region.

Proof. Take a gas particle from region (4.1). We will denote the coordinates and velocity of the chosen particle by $(x_1(\tau), x_2(\tau))$ and $(u_1(\tau), u_2(\tau))$ respectively. The coordinate x_2 of the point E_1 is equal to $f_1(\tau)$, and the velocity of the gas particle at the point E_1 is $(0, f^1{}_1(\tau))$. We will assume that, at a certain instant of time $\tau < 0$ the particle $(x_1(\tau), x_2(\tau))$ leaves the volume (4.1), i.e.

$$x_2(\tau) = f_1(\tau) \tag{4.2}$$

$$u_2(\tau) > f_1(\tau),\ u_1(\tau) \geq 0 \tag{4.3}$$

For the relation between the velocities given by (4.3), it follows from the property of monotonic decrease that $x_2(\tau) < f_1(\tau)$; this contradicts assumption (4.2). Consequently, the point must remain in region (4.1).

Assertion 2. At the final instant of time ($\tau_* = 0$), the curvilinear part of the piston (line DE_1) coincides with the section D_*E_{1*} lying on the straight line $x_2 = r_*$.

Proof. It follows from Assertion 1 that the curvilinear part of the piston is situated in region (4.1); in particular, at the final instant of time for points of the line DE_1 the condition $x_2 \leqslant r_*$ is satisfied, where r_* is the coordinate x_2 of the point E_{1*}. We will prove that when $\tau = \tau_*$ for points on the line DE_1 the equality $x_2 = r_*$ is satisfied. We will assume the contrary, i.e. that for part of the line DE_1 the strict inequality $x_2 < r_*$ is satisfied. It follows from (3.2) that the velocity of these particles is infinite; then the value of the density is also infinite. Hence, in a certain finite volume, adjoining the piston, at the final instant the density will be infinite, which is impossible for a finite mass of gas. Consequently, the last assumption is untrue and the piston coincides with the section D_*E_{1*}.

In Fig. 2 we show compressing pistons at a certain $\tau_1 \in (-1, \tau_*)$. The straight lines AF_1 and BF_2 are drawn through the sections E_1F_1 and E_2F_2 respectively. It follows from Assertion 1 that for gas particles in the region DAE_1, when $\tau \in [\tau_1, \tau_*]$ the inequality $x_2(\tau) \leqslant f_1(\tau)$ is satisfied. In particular, according to Assertion 2 $x_2(\tau_*) = f_1(\tau_*) = r_*$. Hence, at the final instant this part of the gas contracts to the section

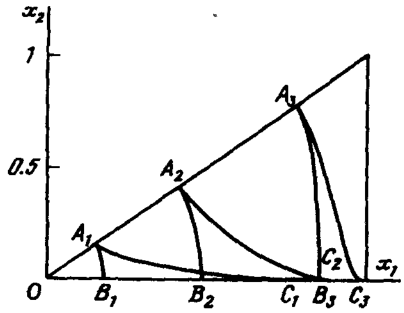

Fig. 2.

D_*E_{1*}. Similarly, region DBE_2 contracts to the section D_*E_{2*}. The region $DACB$—the intersection of regions DAE_1 and DBE_2—then contracts to the point D_*. The form of the regions which contract to a point and to a section can be found approximately by numerical integration of the equations of particle motion (3.3). The results of calculations for $r_* = r_i$ ($r_i = 0.3, 0.6$ and 0.9) are shown in Fig. 2. The regions $A_iB_iC_i$ contract to the section D_*E_{1*}, and regions OA_iB_i contract to the point D

The degree of velocity cumulation. We will now consider the case of the interaction between two similar Riemann waves. The gas flow pattern will be symmetrical about the straight line OS_*, for gas particles in the region DE_1S, and the velocity vector (u_1, u_2) belongs to the set defined by the inequalities

$$0 \leqslant u_1 < \infty, \quad u_2 > hu_1 \tag{4.4}$$

We will obtain the asymptotic form of the velocity increase at the point D as $\tau \to 0$. For a point moving along the axis of symmetry $x_1 = hx_2$, the components of the velocity vector are related by the equation $u_1 = hu_2$. Then

$$x_2 = r_* + (1 + h^2 u_2)\tau + g_2(au_2), \quad a = \delta + h \tag{4.5}$$

We differentiate (4.5) with respect to time and solve for $d\tau/du_2$; as a result we obtain the linear equation

$$\frac{d\tau}{du_2} = \frac{h^2}{(1-h^2)u_2 - 1}\tau + \frac{ag_2'(au_2)}{(1-h^2)u_2 - 1}$$

which is integrated in quadratures

$$\tau = (I(u_2) - k)\left(u_2 + 1 + \frac{1}{\beta}\right)^{1/\beta} \tag{4.6}$$

$$I(u_2) = -a\left(1 + \frac{1}{\beta}\right)\int_0^{u_2} g_2'(au_2)\left(u_2 + 1 + \frac{1}{\beta}\right)^{-1-1/\beta} du_2$$
$$\left(k = -\tau_0\left(1 + \frac{1}{\beta}\right)^{-1/\beta}, \quad \beta = \frac{2-2\gamma}{\gamma+1}\right) \tag{4.7}$$

(τ_0 is the instant when the particle begins to move). The integral $I(u_2) \leqslant k$, since otherwise the function τ becomes positive.

We will prove that $\lim_{u_2 \to \infty} I(u_2) < k_D$, where k_D is the value of the constant k for a particle at the point D. Since the region $DACB$ (Fig. 1) contracts to the point D_*, the velocity of the point C increases without limit. We have $\lim_{u_2 \to \infty} I(u_2) \leqslant k_C$, where k_C is the value of the constant k for the point C. The point D begins to move earlier than the point C, and hence it follows from the definition of the constant k that $k_D > k_C$. Hence, the limit $0 < \lim_{u_2 \to \infty} I(u_2) \leqslant k_C < k_D$ holds; consequently, the first factor on the right-hand side of (4.6) does not vanish as $u_2 \to \infty$. Then, according to (4.6)

$$\tau(u) = O\left(u^{\frac{1}{\beta}}\right), \quad u = O((-\tau)^{\beta}) \tag{4.8}$$

where u is the value of the velocity of the point D, i.e. the degree of cumulation, as in the case of self-similar compression, is equal to β.

An estimate of the optical thickness in the direction of the line of symmetry. We will introduce the function

$$H(\tau,\ C(\tau),\ E(\tau,\ \varphi)) = l_{CE}(\tau) = \int_{CE} \rho ds, \quad C(\tau)$$

the law of motion of a certain point in the region $DE_1F_1S_*F_2E_2$. The law of motion of the optical centre ($C(\tau)$) may not correspond to the trajectory of a certain gas particle. The angle φ specifies the direction of the section CE and the point E lies on the boundary of the region occupied by the gas. The optical thickness is then defined as $l(\tau) = \max_{C(\tau)} \min_{\varphi} H$.

We were unable to determine the value of the optical thickness analytically, so we obtained estimates of H for different directions φ. The optical thickness along the section DS has the form

$$l_{DS}(\tau) = \int_{DS} \rho ds = \int_{x_{2D}}^{x_{2S}} \rho(x_1(x_2),\ x_2,\ \tau) b dx_2 = b \int_{u_D}^{0} \rho(u_1(u_2),\ u_2)\frac{dx_2}{du_2} du_2, \quad b = \sqrt{1+h^2}$$

By (4.5)

$$\frac{dx_2}{du_2} = h^2\tau + ag_2'(au_2), \quad \rho(u_2 h,\ u_2) = (1+h^2u_2)^{\sigma}$$

and we obtain

$$\frac{1}{b} l_{DS}(\tau) = I_1 + I_2$$

$$I_1 = -\int_0^u (1+h^2u_2)^{\sigma} h^2 \tau du_2 = h^2\tau \int_0^u (1+h^2u_2)^{\sigma} du_2$$

$$I_2 = -\int_0^u (1+h^2u_2)^{\sigma} ag_2'(au_2) du_2$$

Here u is the value of u_2 for the point D at the instant τ. Taking into account the fact that $\tau(u) = O(u^{1/\beta})$, we obtain the estimate

$$\int_0^u (1+h^2u_2)^{\sigma} du_2 = O(u^{\sigma+1}), \quad u \to \infty$$

Hence, the integral I_1 is a quantity $O(u^{-1/\beta})$ as $u \to \infty$ and a quantity $O(\tau^{-1})$ as $\tau \to \infty$. It follows from the convergence of integral (4.7) that the order of increase of the quantity I_2 is less than $O(u^{-1/\beta})$. Hence

$$\frac{1}{b} l_{DS}(\tau) = I_1 + I_2 = O\left(\frac{1}{\tau}\right)$$

This asymptotic form of the increase in the optical thickness was obtained for the self-similar case in [4]. This estimate remains true for the optical thickness along any section AB, taken along the line of symmetry, if estimate (4.8) holds for the velocities of points A and B.

An estimate of the optical thickness in a direction different from the direction of the line of symmetry.

Assertion 3. At a certain instant of time $\tau_1 < 0$ on the section DS we choose a gas particle whose coordinates and velocity we will denote by (x, y) and (u, v). We will choose the notation (x_1, x_2) and (u_1, u_2) for the coordinates and velocity of a certain particle, chosen in the region DE_1S. Then (1) if

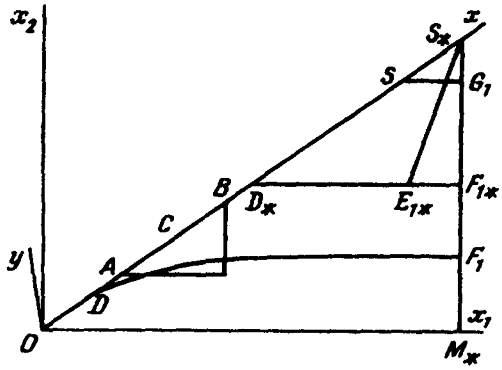

Fig. 3.

$x_1(\tau_1) < x(\tau_1)$, then, for $\tau > \tau_1$, we will have $x_1(\tau) < x(\tau)$; (2) if $y(\tau_1) < x_2(\tau_1)$, then, for $\tau > \tau_1$, we will have $y(\tau) < x_2(\tau)$.

Proof We will assume that, at a certain instant τ

$$x_1(\tau) = x(\tau) \tag{4.9}$$

This means that $u_1(\tau) \geqslant u(\tau)$. It follows from (4.4) that $u_2 > u_1/h \geqslant u/h = \upsilon$. Then, according to the properties of monotonic decrease $x_1(\tau) < x(\tau)$, which contradicts assumption (4.9). The first part of the assertion is proved. The second part is proved in a similar way.

We will introduce a system of coordinates Oxy in which the Ox axis is directed along the straight line OS. At a certain instant of time we choose two gas particles A and B on the Ox axis such that their coordinates are subject to the condition

$$x_D < x_A < x_B \tag{4.10}$$

and estimate (4.8) holds for the velocity of point B (Fig. 3). We will further consider the volume of gas bounded by the straight lines OS, $x_2 = x_{2A}(\tau)$ and $x_1 = x_{1B}(\tau)$, $x_{2A}(\tau)$ is the x_2 coordinate of point A and $x_{1B}(\tau)$ is the x_1 coordinate of point B. We will denote this region by $\Omega(\tau)$. It follows from Assertion 3 that the mass of gas enclosed in the volume $\Omega(\tau)$ does not decrease. This mass of gas at an arbitrary instant of time can be defined by the formula

$$m = \int\limits_{\Omega(\tau)} \rho(x,\ y,\ \tau)dxdy = \int\limits_{x_A}^{x_B}\left(\int\limits_{Y(x)}^{0} \rho(x,\ y,\ \tau)dy\right)dx = \int\limits_{x_A}^{x_B} l(x,\ \tau)dx \tag{4.11}$$

(we have changed from the double integral to a repeated integral). Here $l(x, \tau)$ is the optical thickness in the direction of the straight line Oy. By the theorem of the mean the last term in the chain of equalities (4.11) is equal to $rl(\xi, \tau)$, $x_A \leqslant \xi \leqslant x_B$, $r = x_B - x_A$. Consequently, $l(\xi, \tau) = m/r$. The following estimate of the quantity r follows from (4.5) and (4.10)

$$r \leqslant (r_* - x_{2D})h = hu_2\tau\left(\frac{1}{u_2\tau} + h^2 + \frac{g_2'(au_2)}{u_2\tau}\right) \tag{4.12}$$

where u_2 is the u_2 component of the velocity vector of the point D at the instant of time τ. By (4.8) the estimate $O(u_2{}^{1+1/\beta})$ holds for $u_2\tau$ as $u_2 \to \infty$ and $O((-\tau)^{-h^2})$ as $\tau \to 0$. It follows from the convergence of the integral in (4.7) that $\lim_{u_2\to\infty} g'_2(au_2)/(u_2\tau) = 0$. It then follows from (4.5) that the asymptotic form of the quantity r is determined by the factor $u_2\tau = O((-\tau)^{-h^2})$. We obtain the following estimate for the optical thickness

$$l(\xi,\ \tau) = m/r = O((-\tau)^{-h^{-2}}) \tag{4.13}$$

We will take the point $C = (\xi, 0)$ as the optical centre (in the Oxy system). Then

$$l_{DS} > l_{CD} \geqslant l_{DA}, \quad l_{DS} > l_{CS} \geqslant l_{BS}$$

whence it follows that the following limits hold

$$\frac{c_1}{\tau} < l_{CD} < \frac{c_2}{\tau}, \quad \frac{c_3}{\tau} < l_{CS} < \frac{c_4}{\tau}$$

where $c_i (i = 1, \ldots, 4)$ are positive constants. In this case the optical thickness in the direction of the straight line Oy is not less than $l(\xi, \tau)$, for which estimate (4.13) holds. Similar results were obtained for the self-similar case in [4].

An estimate of the energy costs. The work of the moving piston goes to increasing the amount of kinetic energy E_k and internal energy E_i of the gas. By definition

$$E_k = \int_V \frac{u_1^2 + u_2^2}{2} \rho dV$$

while for the equation of state considered

$$E_i = E_{i1} \int_V p dV + E_{i2}, \quad E_{i1}, \; E_{i2} = \text{const}$$

We will denote the length of the velocity vector by u. The following asymptotic relations hold

$$r = c^\sigma = O(u^\sigma), \quad p = A\rho^\gamma = O(u^{\sigma+2}), \quad u \to \infty$$

Consequently, the integrands in the formulae for the potential and kinetic energy as $u \to \infty$ are of the same order: $O(u^{\sigma+2})$. Hence, to estimate the energy costs it is sufficient to estimate the value of the kinetic energy. To do this we will estimate the velocity values in the neighbourhood of the point D in Fig. 1.

Assertion 4. We will take a gas particle on the line of symmetry, for whose velocity estimate (4.8) is satisfied; we will denote the chosen particle by A. Consider the mobile region

$$x_2 < x_1 / h, \quad x_1 < x_{1A} \tag{4.14}$$

Positive constants k_1 and k_2 exist such that for any point of region (4.14), for any τ we have $k_1 u > u_D > k_2 u$, where u_D and u are the velocity values of the point D and of points of the region (4.14), respectively.

Proof. We will assume the opposite: for any constants k_1 and k_2 and instant of time τ and a point from the region (4.14) exist such that $k_1 u \leqslant u_D$ or $u_D \leqslant k_2 u$. Estimate (4.8) holds for the velocity of points A and D; consequently, a constant $k > 0$ exists for which $u_D < k u_A$, for instants of time close to the final instant. We will take $k_1 = kb$, $k_2 < (3 - \gamma)/2$.

We will assume that in the volume in question there is a particle for which $k_1 u \leqslant u_D$. Then $k_1 u_2 (u_1^2/u_2^2 + 1)^{1/2} < u_{2D} b < k u_{2A} b$ and consequently $u_2 < u_{2A}$, $u_{1A} > u_2 h > u_1$. It follows from the property of monotonic decrease that $x_2 > x_{2A}$. This contradicts the second part of Assertion 3.

We will assume that a particle exists for which $u_D \leqslant k_2 u$, i.e. $(u_{1D}^2 + u_2^2 D)^{1/2} < k_2 (u_1^2 + u_2^2)^{1/2}$. Then $u_{2D} < u_{2D} b (u_1^2 + u_2^2 + 1)^{-1/2} < k_2 u_2$. In addition, $u_1 > 0$. It follows from formulae (3.2) and (4.5) and the property of monotonic decrease that $x_2 < x_{2D}$, which contradicts the second part of Assertion 3.

Then, the following estimate exists for the kinetic energy of the gas enclosed in the volume (4.14)

$$E_k \in \left[\frac{m u_{\min}^2}{2}, \frac{m u_{\max}^2}{2} \right], \quad m = \int_V \rho dV$$

where m is the mass of gas enclosed in the volume (4.14), and $u_{\min}$ and $u_{\max}$ are the minimum and maximum velocity values in this volume. It follows from Assertion 4 that, in the region (4.14), the following estimate holds for the total energy of the gas

$$c_1 u_D^2 < E < c_2 u_D^2, \quad c_1, \ c_2 = \text{const} > 0$$

Taking estimate (4.8) of the value of the velocity of the point D into account we obtain $c_1(-\tau)^{2\beta} < E < c_2(-\tau)^{2\beta}$. The remaining gas volume does not change this estimate since the velocity in this part is less than the velocity of the point D.

The estimate $E = O((-\tau)^{2\beta})$ was obtained in [3] for self-similar compression.

5. UNMATCHED INTERACTION BETWEEN SELF-SIMILAR RIEMANN WAVES

The gas flow pattern will be symmetrical about the straight line OS_*, so we will henceforth consider only half of the prism OS_*M_*. The flow in a self-similar Riemann wave (the trapezium $E_1F_1G_1S$, Fig. 1) is defined by the formulae

$$\Theta_R = \sigma + u_2, \quad x_2 = r_* + (u_2 + \sigma^{-1}\Theta_R)\tau$$

In the region DE_1S the gas flow is a double self-similar wave, in which the coordinates of the particles and the components of the velocity vector are related by the equations

$$x_i = (u_i + \sigma^{-1}\Theta\Theta_i)\tau + g_i, \quad g_1 = r_* \operatorname{ctg}\alpha, \quad g_2 = r_* \tag{5.1}$$

The function $\Theta(u_1, u_2)$ is found from Eq. (2.3). The following condition corresponds to a double wave adjoining a simple wave, in which $u_1 = 0$.

$$\Theta(0, \ u_2) = \Theta_R(u_2) \tag{5.2}$$

The symmetry about the straight line $x_1 = x_2 \operatorname{ctg} \alpha$ (the straight line OS_*) leads to satisfaction of the impermeability condition $u_1 = u_2 \operatorname{ctg} \alpha$ on the straight line. From (5.1) we obtain that the impermeability condition to occur along the straight line OS_* corresponds to the second boundary condition

$$\Theta_1(u_2 \operatorname{ctg}\alpha, \ u_2) = \Theta_2(u_2 \operatorname{ctg}\alpha, \ u_2)\operatorname{ctg}\alpha \tag{5.3}$$

The equation of double waves must be solved in the set

$$0 \leqslant u_1 \leqslant u_2 \operatorname{ctg}\alpha, \quad 0 \leqslant u_2 < \infty \tag{5.4}$$

Moreover, to solve (2.3) numerically it is necessary to determine the quantity $\Theta_1(0, u_2)$. It follows from (5.2) that

$$\Theta_2(0, \ u_2) = 1, \quad \Theta_{22}(0, \ u_2) = 0 \tag{5.5}$$

Substituting these quantities into Eq. (2.3) and introducing the notation

$$x = \Theta(0, \ u_2) = \sigma + u_2, \quad y(x) = \Theta_1^2(0, \ u_2)$$

we obtain the following equation, which is satisfied on the straight line $u_1 = 0$ (we have in mind a straight line in the plane of the hodograph, Ou_1u_2)

$$x dy / dx + (1 - \sigma)y = -(1 + \sigma)$$

solving which, we obtain

$$\Theta_1(0, \ u_2) = [k(\sigma + u_2)^{\sigma - 1} + h^2]^{1/2}, \quad k = \text{const} \tag{5.6}$$

Substituting $u_1 = u_2 = 0$ into condition (5.3) we obtain $\Theta_1(0, 0) = \Theta_2(0, 0) \operatorname{ctg} \alpha = \operatorname{ctg} \alpha$. From this equation we find

$$k = \sigma^{1-\sigma}(\operatorname{ctg}^2 \alpha - h^2)$$

It follows from (5.2) that $\Theta_2(0, u_2) = 1$. By substituting this quantity into the equation of the direction of the characteristics

$$\frac{du_1}{du_2} = \frac{\Theta_1\Theta_2 \pm \sqrt{\Theta_1^2 + \Theta_2^2 - 1}}{1 - \Theta_1^2}$$

it can be shown that the straight line $u_1 = 0$ is a characteristic (at least, sections of this straight line on which $\Theta_1 \geqslant 0$).

Consider the case when $\alpha < \alpha_0$. We need to solve Eq. (2.3), which satisfies boundary conditions (5.2), (5.3) and (5.6). Hence, we have set up a mixed problem with boundary conditions on the characteristic straight line $u_1 = 0$ and with additional conditions along the line, which is not a characteristic. This problem was solved numerically by the method of characteristics in [4].

If $\alpha > \alpha_0$, then $k < 0$, and for a certain value $u_2 = u_c$ the derivation of $\Theta_1(0, u_c)$ becomes zero. By substituting the values $\Theta_1 = 0$, $\Theta_2 = 1$ into the equation of the direction of the characteristics, it can be shown that the directions of the characteristics of the two families at this point are identical, i.e. when $\alpha > \alpha_0$ Eq. (2.3) is not hyperbolic over the whole region (5.4). It is only possible to obtain the degeneracy line, specified by the relation $\Theta_1^2 + \Theta_2^2 = 1$, numerically by solving Eq. (2.3) by the method of characteristics.

A shock wave on the boundary of a simple and a double wave. At the instant when the shock wave is formed the Jacobian of transformation (5.1) becomes equal to zero. Consider the expression

$$J(u_1, u_2) = \frac{1}{(t-1)}\frac{D(x_1, x_2)}{D(u_1, u_2)} =$$
$$= \left(1 + \frac{\Theta_1^2 + \Theta\Theta_{11}}{\sigma}\right)\left(1 + \frac{\Theta_2^2 + \Theta\Theta_{22}}{\sigma}\right) - \left(\frac{\Theta_1\Theta_2 + \Theta\Theta_{12}}{\sigma}\right)^2$$

The line $u_1 = 0$ corresponds to the line where the double wave joins the simple wave in the hodograph plane, along which

$$J(0, u_2) = \left(1 + \frac{1}{\sigma}\right)\left(1 + \frac{\Theta_1^2 + \Theta\Theta_{11}}{\sigma}\right) - \left(\frac{\Theta_1\Theta_2 + \Theta\Theta_{12}}{\sigma}\right)^2$$
$$\Theta_{12}(0, u_2) = k(\sigma - 1)\Theta^{\sigma-2}(2\Theta_1)^{-1} \qquad (5.7)$$

We will obtain the derivative of $\Theta_{11}(0, u_2)$. Differentiating Eq. (2.3) with respect to the variable u_1, we obtain

$$\Theta_1((1-\Theta_2^2)\Theta_{11} + 2\Theta_1\Theta_2\Theta_{12} + (1-\Theta_1^2)\Theta_{22}) +$$
$$+\Theta[(1-\Theta_2^2)\Theta_{111} + 2\Theta_1\Theta_{12}\Theta_{12} + 2\Theta_1\Theta_2\Theta_{112} - 2\Theta_1\Theta_{11}\Theta_{22} + \qquad (5.8)$$
$$+(1-\Theta_1^2)\Theta_{122}] + (1-\sigma)(2\Theta_1\Theta_{11} + 2\Theta_2\Theta_{12}) = 0$$

We will introduce the notation

$$x = \Theta(0, u_2),\ y(x) = \Theta_{11}(0, u_2),\ y'(x) = \Theta_{112}(0, u_2)$$
$$a(x) = \Theta_1(0, u_2),\ a'(x) = \Theta_{12}(0, u_2),\ a''(x) = \Theta_{122}(0, u_2)$$

Taking into account the fact that equalities (5.5) are satisfied for the straight line u_1, we can write Eq. (5.8) in the form

$$y' = \frac{\sigma - 1}{x}\left(y - \frac{a'}{a}\right) - \frac{aa'}{x} - (a')^2 - \frac{(1-a^2)a''}{2a} \qquad (5.9)$$

A solution can only be obtained in terms of elementary functions for certain values of the adiabatic exponent; in the remaining cases this linear differential equation was solved numerically. For $\gamma = 5/3$ the general solution has the form

$$\Theta_{11}(0, u_2) = -\frac{\Theta_R}{16(k\Theta_R^2+2)} \times \begin{cases} f, & \alpha < \alpha_0 \\ g, & \alpha > \alpha_0 \end{cases}$$

$$f = (17\sqrt{2k^3}\Theta_R \operatorname{arctg}(\sqrt{k/2}\Theta_R) - 32\Theta_R l)(k\Theta_R^2+2) + 2k(k\Theta_R^2+4)$$

$$g = -(17\sqrt{-2k^3}\Theta_R \operatorname{arcth}(\sqrt{-k/2}\Theta_R) + 32\Theta_R l)(k\Theta_R^2+2) - 2k(k\Theta_R^2+4), \quad l = \mathrm{const}$$

The constant l can be obtained from condition (5.3), by differentiating which along the straight line $u_2 = u_1$ tgα, we obtain relations that are satisfied on this straight line

$$\frac{d\Theta_n}{du_1} = \Theta_{1n} + \operatorname{tg}\alpha\Theta_{n2}, \quad n = 1, 2; \quad \frac{d\Theta_2}{du_1} = \operatorname{tg}\alpha\frac{d\Theta_1}{du_1}$$

Hence we obtain the value of the derivative $\Theta_{11}(0, 0) = (\operatorname{ctg}\alpha - \operatorname{tg}\alpha)\Theta_{12}(0, 0)$. Substituting the value obtained for the derivative $\Theta_{11}(0, u_2)$ into (5.7), it can be shown that if the angle $\alpha \neq \alpha_0$, a certain value $u \in (0, +\infty)$ exists for which $J(0, u) = 0$. Hence, a shock wave occurs when the velocities of the gas particles and their density are finite. When $\alpha > \alpha_0$ a shock wave is formed up to the line of parabolic degeneration of the equation of the double waves. The velocity value for which a strong discontinuity occurs is less than the corresponding value for $\alpha < \alpha_0$. From the point of view of obtaining large local densities it is interesting to consider the case when $\alpha < \alpha_0$. Numerical solution of Eq. (5.9) for different values of α and γ leads to the same conclusions.

The results of numerical calculations. After finding the function Θ the gas flow in the double-wave region is defined by (5.1). We must now indicate the law of the piston motion which ensures such compression. This problem reduces to integrating the system

$$dx_1/dt = u_1, \quad dx_2/dt = u_2$$

The components of the velocity vector are found from (5.1). The initial conditions correspond to the conditions for the double wave to approach a simple wave at the instant $t = t_0$ at the point E_1F (Fig. 1)

$$u_1(t_0) = 0, \quad u_2(t_0) = u_R(t_0) = \sigma(1-t_0)^{-1/(1+\sigma)} - \sigma$$

where u_R is the velocity of the straight part of the piston (section E_1F_1, Fig. 1).

The law of the piston motion was found numerically. We carried out calculations which indicate what degree of compression can be obtained when self-similar Riemann waves interact. Thus, for $\gamma = 5/3$ (a monatomic gas), the maximum local degree of compression (at point D in Fig. 1) at the instant when a shock wave occurs is equal to 19,000, for $\alpha = \pi/6$, but the density increases only by a factor of 1450 for $\alpha = \pi/9$. The closer the value of the angle to the matched value the greater the degree of compression that can be achieved at the instant when the strong discontinuity occurs.

Calculations were carried out on meshes which differed in the choice of mesh width on the straight line $u_1 = 0$ in the hodograph plane, with a minimum mesh width of 0.005. Good agreement between the results of calculations carried out on meshes with mesh widths differed by a factor of two were obtained.

Figures 4–7 were drawn using calculations of the compression of a gas with an adiabatic exponent of $\gamma = 5/3$. In Figs 4 and 5 we show fragments of the fields of the characteristics in the plane of the self-similar variables $\xi_i = (x_i - g_i)/\tau$, and also the positions in this plane (at the instant when the shock wave occurs) of the mobile pistons and the boundaries of propagation of the sound perturbation. The same notation as in Fig. 1 is used, namely, E_1F_1 is the rectilinear part of the piston and $E_1F_1G_1S$ is the region in which the gas flow is a Riemann wave. A shock wave occurs on the piston surface, at the boundary of the straight line and the rectilinear part (point E_1 in Fig. 1). At point E_1 the characteristics of one family intersect without touching, and the corresponding characteristics in the hodograph plane do not intersect. The mobile piston is represented incompletely since the remaining part of the piston is considerably elongated along the line of symmetry and practically merges with it.

The density distribution along the straight line OS_* (at certain fixed instants of time, the distance x is measured from the point S_* and the unperturbed gas density is ρ_0) is shown in Fig. 6 for $\alpha = \pi/6$ at $t = 0.979$, 0.988 and 0.992, and in Fig. 7 for $\alpha = \pi/9$ at $t = 0.907$, 0.947 and 0.963 (curves 1, 2 and 3, respectively).

Concluding notes. Asymptotic estimates for the non-self-similar case show that important asymptotic properties of the interaction of self-similar Riemann waves may be retained when there is a considerable

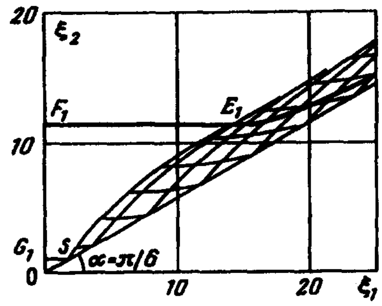

Fig. 4.

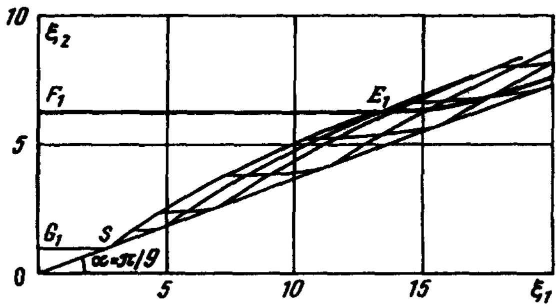

Fig. 5.

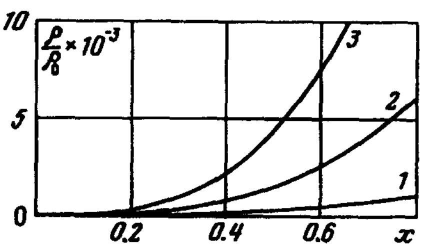

Fig. 6.

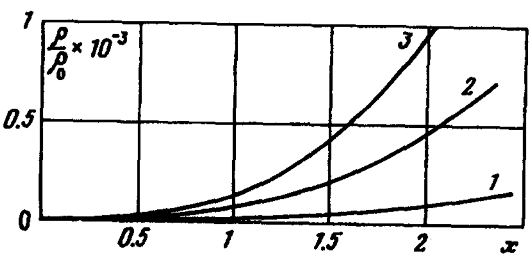

Fig. 7.

change in the law of motion of the compressing piston, which indicates partial stability of this process.

However, the interaction of self-similar Riemann compression waves at an angle which differs from the matched angle does not enable an unlimited increase in the gas density to be obtained without the formation of shock waves. Although numerical calculations have shown that the maximum local density may increase by a factor of tens of thousands compared with the unperturbed state, to solve the problem of unlimited unshocked compression it is necessary to invoke new classes of flows [2].

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